MATH 3235 Probability Theory

11/10/22

Flip a coin

H $+1$

T -1

$2N$ Times N large

$S_N = \sum_{i=1}^{2N} X_i$ X_i individual coin flip.

$m \simeq \delta\sqrt{N}$

Chebychev

$$\mathbb{P}(|S_N| > t) \leq \frac{\mathbb{E}(S_N^2)}{t^2}$$

$$= \frac{2N}{t^2}$$

$t \simeq \sqrt{N}$

$t = 2\sqrt{N}$

$$\mathbb{P}(|S_N| > t) \leq \frac{1}{2}$$

$$m = \delta\sqrt{N}$$

$$\mathbb{P}\left(2m - 1 \leq S_N \leq 2m + 1\right) =$$

$$\binom{2N}{N-m} 2^{-2N}$$

$$N! \simeq \sqrt{2\pi N}\; N^N e^{-N}$$

$$\binom{2N}{N-m} = \frac{(2N)!}{(N-m)!\,(N+m)!}$$

$$\frac{\sqrt{2\pi 2N}}{\sqrt{2\pi(N-m)}\;\sqrt{2\pi(N+m)}} \simeq \frac{\sqrt{2}}{\sqrt{2\pi N}}$$

$$m = \delta\sqrt{N}$$

$$\frac{(\cancel{2}N)^{2N}\;\cancel{2}^{-2N}\;\cancel{e^{-2N}}}{(N-m)^{N-m}\;\cancel{e^{-(N-m)}}\;(N+m)\cancel{e^{-(N+m)}}} =$$

$$\frac{(N-m)^{-(N-m)}\,(N+m)^{-(N+m)}}{N^{-2N}} =$$

$$\left(1-\frac{m}{N}\right)^{-(N-m)}\left(1+\frac{m}{N}\right)^{-N+m} =$$

Remark:

$$\left(1+\frac{1}{N}\right)^{N} \to e$$

$$= \exp\left(-(N-m)\log\left(1-\frac{m}{N}\right) - (N+m)\log\left(1+\frac{m}{N}\right)\right) =$$

$$= \exp\left(-N\left(\left(1-\frac{m}{N}\right)\log\left(1-\frac{m}{N}\right) + \left(1+\frac{m}{N}\right)\log\left(1+\frac{m}{N}\right)\right)\right)$$

$$\frac{m}{N} \approx \frac{\delta}{\sqrt{N}} \qquad \text{small}$$

$$\log(1+x) = x - \frac{x^2}{2} + \dots$$

$$\exp\left(-N\left(\left(1-\frac{\delta}{\sqrt{N}}\right)\left(-\frac{\delta}{\sqrt{N}}-\frac{\delta^2}{2N}\right)+\right.\right.$$

$$\left.\left.\left(1+\frac{\delta}{\sqrt{N}}\right)\left(\frac{\delta}{\sqrt{N}}-\frac{\delta^2}{2N}\right)\right)\right) =$$

$$\exp\left(-N\left(\frac{\delta^2}{N}+\frac{\delta^4}{N^2}\quad\dots\right)\right) =$$

$$\exp\left(-\delta^2\right)$$

All Together

$$\frac{1}{\sqrt{2\pi\,\frac{1}{2}}}\; e^{-\frac{\delta^2}{2\,\frac{1}{2}}}\; \frac{1}{\sqrt{N}} \quad d\delta$$

Central limit Theorem

X_i be r.v. i.i.d. with

$\mathbb{E}(X_i) = 0$ $\quad$ $var(X_i) = 1$

$$Z_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i$$

$$Z_N \to ??$$

$$U = \begin{cases} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{cases}$$

$$X_i = (-1)^i U$$

Z_N $\quad$ F_N $\quad$ The c.d.f. of Z_N

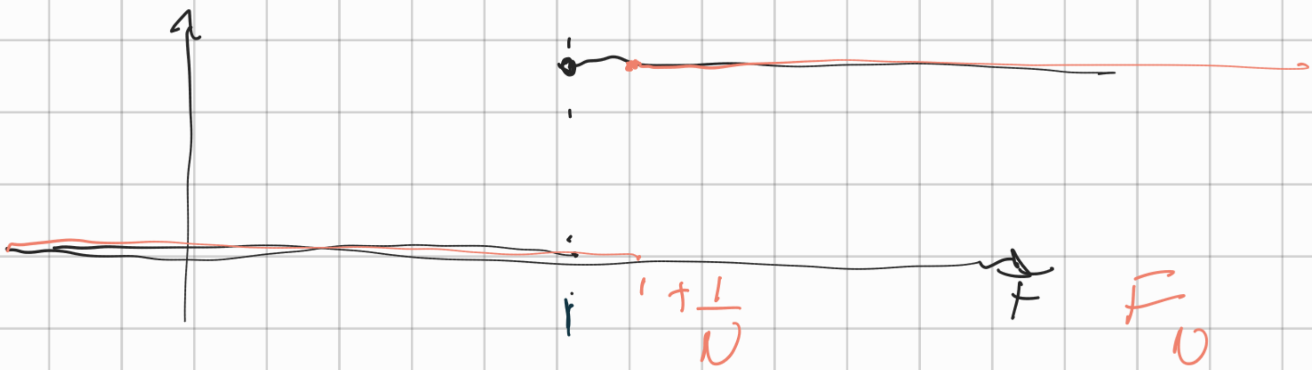
Z $\quad$ F $\quad$ The c.d.f of Z

$$Z_N \Rightarrow Z \quad , f$$

$F_N(z) \to F(z) \quad \forall z$ where F is cont.

$Z_N \quad 1 + \frac{1}{N} \quad$ prob 1

$Z \quad 1 \quad$ prob 1



$F_N(1) = 0 \qquad F(1) = 1$

$Z_N \to Z$ in dist.

$$\mathbb{P}(Z_N \le z) \to \mathbb{P}(Z \le z)$$

for every z such that

$\mathbb{P}(Z = z) = 0$

Geometric r.v. with par $p = \frac{\lambda}{N}$

X_N let X an exp r.v. with

par λ

$$\frac{X_N}{N} \Rightarrow X$$

$$\mathbb{P}\left(\frac{X_N}{N} > t\right) = \left(1 - \frac{\lambda}{N}\right)^{Nt} \to e^{-\lambda t} =$$

$$\mathbb{P}(X > t).$$

Geometric $\to$ Exp

Binomial $\to$ Normal

Continuity Theorem

Let Z_N be r.v. with

char. fun. $\phi_N(t)$. Assume

$\phi_N(T) \to \phi(t) \quad \forall t \in \mathbb{R}.$

$$\Downarrow$$

Then $Z_N \Rightarrow Z$.

X_i $\quad \phi_X(t) = \left(1 + i\mathbb{E}(X)t - \mathbb{E}(X^2)\frac{t^2}{2} + \dots\right)$

$\mathbb{E}(X) = 0 \qquad \mathrm{var}(X) = 1$

$$\phi_{\frac{X}{\sqrt{N}}} = \left(1 - \frac{t^2}{2N} + O\left(\frac{t^3}{N^{3/2}}\right)\right)$$

$$Z_N = \sum_{i=1}^{N} \frac{X_i}{\sqrt{N}}$$

$$\phi_{Z_N}(t) = \left(\phi_{\frac{X}{\sqrt{N}}}(t)\right)^N =$$

$$\left(1 - \frac{t^2}{2N} + O\left(\frac{t^3}{N^{3/2}}\right)\right)^N =$$

$$e^{-\frac{t^2}{2}} = \phi_Z(t) \quad \text{with} \quad Z \sim \mathcal{N}(0,1)$$

If X_i are i.i.d. r.v. with

$$\mathbb{E}(X_i) = \mu \qquad var(X_i) = \sigma^2$$

Calling

$$Z_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{X_i - \mu}{\sigma}$$

$Z_N \Longrightarrow Z$ where Z in $\mathcal{N}(0,1)$.

X_i are i.i.d.

$$\sum_{i=1}^{N} X_i \simeq \mathcal{N}\left(N\mu, N\sigma^2\right)$$

$$\frac{1}{N}\sum_{i=1}^{N} X_i \simeq \mathcal{N}\left(\mu, \frac{\sigma^2}{N}\right)$$

$$a_N \to b_N$$

$$\not\Downarrow$$

$$N a_N \to N b_N$$

X_i are The result of measurement with an apparatus. I Know

$E(X_i) = \mu$ is The True value

$Var(X_i) = \sigma^2$

I Take N measurement x_i

$i = 1 \dots N$.

$$\bar{x} = \frac{1}{N} \sum_i x_i$$

Find δ such That

$$\bar{x} - \delta \le \mu \le \bar{x} + \delta$$

with prob 95%.

I need To Take $\delta = 1.96 \frac{\sigma}{\sqrt{N}}$

That with prob. 0.95

$$\bar{x} - 1.96 \frac{\sigma}{\sqrt{N}} \le \mu \le x + 1.96 \frac{\sigma}{\sqrt{N}}$$

if N is large.

0.5 prob of H.

0.01 ,

0